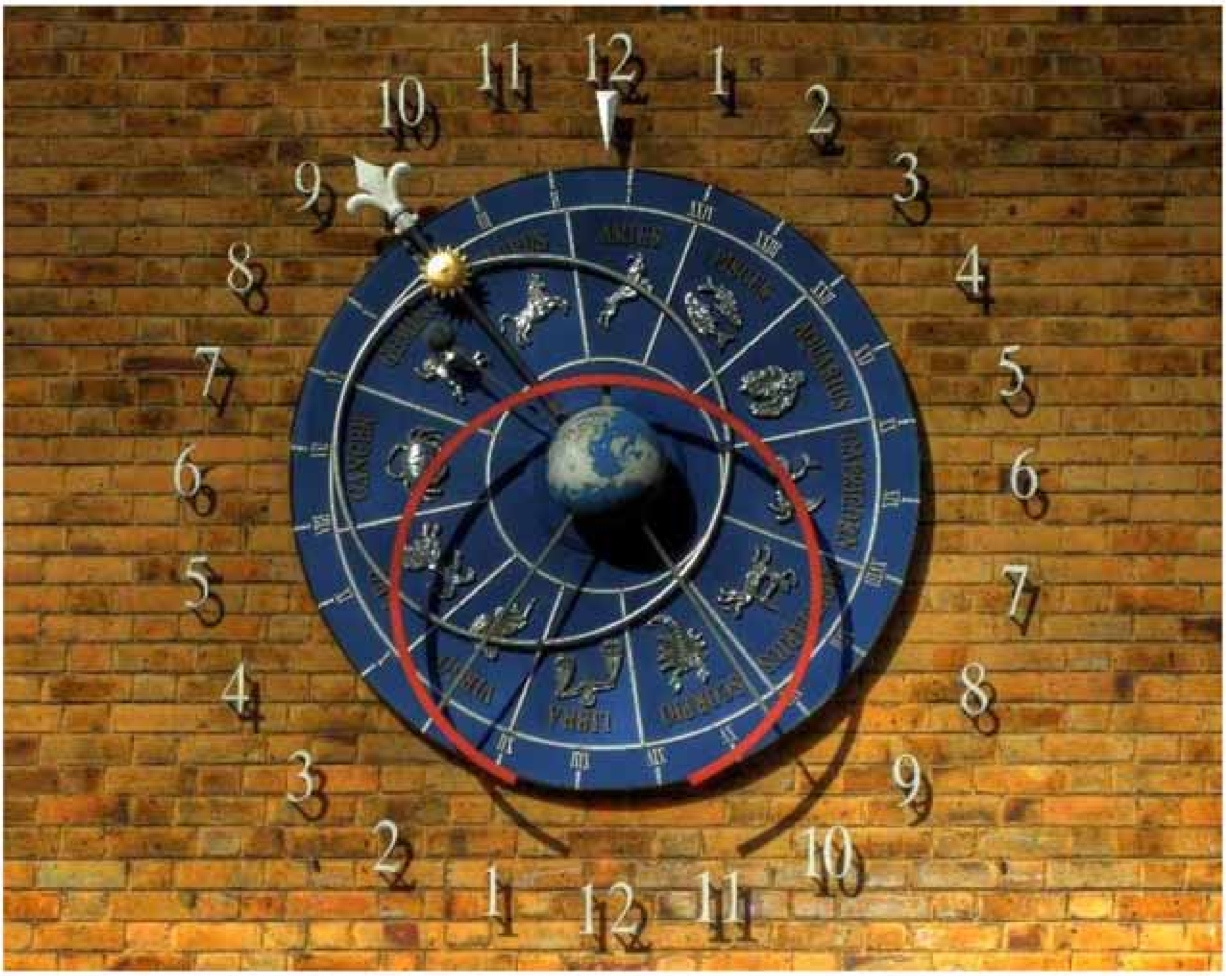
# GIREP-EPEC & PHEC 2009 INTERNATIONAL CONFERENCE

August 17-21, University of Leicester, UK

# PHYSICS COMMUNITY AND COOPERATION

Volume 2



## SYMPOSIUM

### MATHEMATIZATION IN PHYSICS LESSONS: PROBLEMS AND PERSPECTIVES

Gesche Pospiech, Ricardo Karam, Esther Bagno, Edward F. Redish, Ulrike Böhm, Maurício Pietrocola, Hana Berger, Bat Sheva Eylon, & Thomas J. Bing

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EDITORS
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# MATHEMATIZATION IN PHYSICS LESSONS: PROBLEMS AND PERSPECTIVES

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## ABSTRACT

The deep interrelations between physics and mathematics, as well as the essential role of this mutual influence for the development of both, can be confirmed by a brief excursion into the history of scientific knowledge. However, in the context of science education, this crucial subject is far from being trivial. Taking into account the complexity of the theme, the main focus of this symposium is to analyze thoroughly the functions and the aims of mathematics in physics lessons.

Naturally, an issue as complex as the *mathematization in physics lessons* has to be approached from multiple perspectives and theoretical frameworks. Therefore, the four presented papers have not only different approaches - cognitive, epistemological, teaching and learning - but also focus their research in different age groups - covering lower secondary school, higher secondary school, first years of university education and graduate physics majors. This rich variety shows that on each level and framework there are specific problems, difficulties or characteristics, but it is also possible to detect several common aspects. During the symposium further research desiderata in this field of physics education were derived and are presented here.

## 1. INTRODUCTION

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Science education or - more specifically - physics education aims at giving the students insight not only into the laws of physics but also into the scientific processes and the generation of physical knowledge. In this connection, besides a qualitative or phenomenological conceptual understanding, the role and the use of mathematics in physics shows relevant for physics teaching. Also from the viewpoint of the nature of science it seems essential that students experience different aspects of the role of mathematics in physics and learn the mathematization process which hence has to have its proper place in education.

In this symposium we have approached this complex issue from several perspectives and frameworks - the epistemological, cognitive and teaching. These frameworks find themselves in the contributing papers with different emphasis and will be outlined in this introduction.

## 2. THE EPISTEMOLOGICAL FRAMEWORK

Several historical examples - such as the development of calculus and its relation to mechanics or the importance of vectorial analysis for the formalization of electromagnetism - highlight the fruitful interplay between physics and mathematics. Considering modern and contemporary physics theories, it is possible to infer that mathematics is so profoundly incorporated that they can no longer be divided into a mathematical part and a conceptual (non-mathematical) one. In some cases, such as the antiparticle prediction, mathematical entities emerge from the abstract formalism and are physically interpreted *a posteriori*.

In spite of this flagrant interdependence, when we focus our analysis in the context of physics education, it is rather common to find mathematics being treated as a mere instrument to quantify empirical relations and to solve standard physical problems. Looking for a equation inside a "mathematical toolbox" and using it to express a particular physical phenomenon or telling the students, after judging having "physically" interpreted a problem, that "from now on it is just mathematics", are classical, yet not unusual, examples of this conception.

This incompatibility of a purely mathematical and a purely physical view is a critical subject in physics education and finding an adequate balance is far from being trivial. We move in a complex interrelation where not only the different understandings and perspectives of the subjects themselves, but also of their didactics, interfere with each other. Aside from the mutual influence between mathematics and physics, it is possible to point out some considerable differences in their methods and goals. While the first is more focused on the structure, providing exact definitions, proofs and generalizations, the latter is mainly concerned with explaining phenomena from the "real" world. According to Feynman "the physicist is always interested in the special case; he is never interested in the general case. He is talking about something; he is not talking abstractly about anything". These differences might cause technical or semantic problems for the students in switching from one view to the other.

Seen from a very basic level, mathematics provides abstract forms whereas physics deals with natural phenomena. The link between these two has to be done by giving meaning to the mathematics and structuring the physical appearances. Both these linking processes - at the heart of the physical method - have to be made clear to students in order to enable them to transgress the border of the subjects: interpreting the "pure" mathematics and formalizing the "dirty" physics. Furthermore the mathematical symbols such as diagrams, equations and formulae serve as a communicative tool in giving clear information on the limitations of e.g. physical laws or additional information. If mathematics were only regarded as a tool for calculating numerical results this important aspect would be neglected.

Physical argumentation is at the heart of doing, understanding and teaching physics. This mainly has a linguistic component. But the corresponding frame is prescribed by the mathematical structure which is an important part of conveying the implications of the laws exactly and is used for predictions. To make clear this interplay - the necessity of giving the mathematics a physical meaning - to the students is an important task in physics education. If the goal is to develop an adequate view on the role of mathematics in physics, a suitable instruction has to be implemented. Naturally, this demands concentrating research efforts in that subject.

## 3. The cognitive framework

Very little is known, also from research in mathematics education, about how students really attack tasks in physics with mathematical elements. This subject is normally governed by general beliefs but not by solid empirical evidence. Students' difficulties often are related to insufficient mathematical skills, to poor transfer ability or to deficient physics knowledge. Less focus is directed on the problem solving strategies the students employ in solving quantitative physical tasks. It is well known that novices and experts differ strongly in their procedures for attacking a given problem. Among teachers there is a strong tendency to provide the novice students with recipes for the process of solution. However, there is no detailed account of the strategies students would use on their own or were able to use. One attempt has been made by Tuminaro & Redish (2007), identifying so-called epistemic games college students might use. Bing & Redish (2009) go along the same line, but with university students. The identification of strategies with younger students may lead to hints where some of the obvious difficulties may have their roots and how they could be addressed.

In analyzing the cognitive processes it has generally to be taken into account that students and teachers both have their own frame of reference from which they construct their lessons (instruction) or from which they conceive the contents of the lesson, respectively. We assume that this framing not only takes place in higher education but even - and perhaps even more influentially - in the very beginning of science or physics lessons. Therefore the way students use mathematics in physics, their ability to cope with the semantics, will be developed by habitude during their school career. This hypothesis motivates especially studies in lower secondary school concerning the role students attribute to mathematization in physics and their corresponding abilities. Hence, it might be important to analyze the teaching-learning processes concerning mathematization already at this early stage. This will be an important condition for fostering the competence of (mathematical) modeling, one of the big aims of physics education. In order to improve students' understanding of physics and the physical method, the role of physical and mathematical models and their interplay has to be clarified for both the teachers and the learners.

## 4. The teaching framework

The teaching aspect is strongly related to the cognitive perspective but focuses more on the implementation. The basis is to design teaching-learning sequences providing a systematic introduction of students into the use of mathematics in physics. Herewith we think that metacognitive activities play an important role. They would have to promote a conceptual understanding of physical formulae and prevent students from blindly applying a set of given mathematical relations to artificial physical situations. It should be possible to identify crucial points for successfully teaching the transfer between physical phenomena and their mathematical description. In the light of the epistemological framework outlined above, the interpretation of formulae on different levels and the development of suitable strategies in problem solving are essential elements in a teaching strategy.

Another important aspect concerns motivation. Whereas it is said the students do not like the mathematization of physics we assume that suitable examples with strong connection to everyday life and the experience of competence might enhance motivation and interest.

## 5. Problems and Perspectives

The goal of this Symposium was to identify research desiderata related to these questions and open questions concerning the use of mathematics in physics lessons.

One general concern is: which important aspects mathematization could contribute to physics education? Could mathematization deepen the understanding of physics, physical concepts and the method of science already at an early stage? Under the assumption that scientific literacy is the core of physics education, students should have an appropriate view on the role of mathematics in physics, what it can achieve and what is beyond the range of it. In order to develop adequate strategies, the first step is to determine the present state at schools on different levels and from different standpoints, taking into account the views and the motivation of students.

In this international symposium it turned out that there are quite different situations because of the school systems such that there are specific questions to be posed and the research adapted accordingly. This mainly concerns the adequate teaching strategies and the age group at which they should start. Accordingly, a curricular question would be: should students acquire mathematical skills previously to their contact with physics, or should the mathematical entities have “physical correspondents” from the beginning? On the other hand there can be mathematization of physics before the use of formulae. How can such elements (e.g. diagrams) be used to prepare the following steps?

Other questions are more related to the teachers’ perspective. What are the differences between the mathematics taught in regular maths classes and the one used by physicists to model phenomena and structure their thought? What are the conceptions of physics’ teachers about the interdependence between mathematics and physics and how can they affect their lessons?

Which kinds of teaching strategies can be implemented to stress the structural role of mathematics in physics? How do students think when they make use of mathematics to solve physical problems and based on which arguments do they justify this use?

Many of these questions were approached in this symposium and many others emerged from the discussions. Our goal is to continue and amplify this fruitful debate in discussion spaces of our community, such as conferences and journals.

## 6. OUTLINE OF THE CONTRIBUTIONS

From a cognitive perspective, Edward Redish and Tom Bing (USA) present some of their recent studies concerning the development of analytical tools, namely epistemic games and frames, which emerged from the analysis of students solving physics problems. Their main goal is to understand how students make use of mathematics to solve physics problems and how they warrant this use.

Facing the challenge of students' understanding of physical formulae, Esther Bagno, Bat Sheva Eylon and Hana Berger (Israel) describe a diagnostic study they have conducted, as well as a classroom activity they have designed, which focuses on improving students' conceptual interpretation of a physical formula.

From an epistemological basis, Ricardo Karam and Mauricio Pietrocola (Brazil) propose a distinction between technical and structural skills concerning the use of mathematics in physics and point out some of the abilities which should be aimed at in physics education in order to develop the students' recognition of the structural role of mathematics in physical thought.

In order to identify students' and teachers' views on the use of mathematics in physics, Gesche Pospiech and Ulrike Böhm (Germany) have conducted a study which enabled them to categorize these conceptions. Some of their main results, concerning students' and teachers' manifestations of the importance of verbal explanations, mathematical formulations and graphical means for their understanding of physics, are presented and discussed.

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# USING MATH IN PHYSICS: WARRANTS AND EPISTEMOLOGICAL FRAMES

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## ABSTRACT

Mathematics is an essential component of university level science, but it is more complex than a straightforward application of rules and calculation. Using math in science critically involves the blending of ancillary information with the math in a way that both changes the way that equations are interpreted and provides metacognitive support for recovery from errors.

We have made ethnographic observations of groups of students solving physics problems in classes ranging from introductory algebra-based physics to graduate quantum mechanics. These observations lead us to conjecture that expert problem solving in physics requires the development of the complex skill of mixing different classes of warrants - the ability to blend physical, mathematical, and computational reasons for constructing and believing a result.

In order to analyze student behavior along this dimension, we have created analytical tools including epistemological framing and resources, (Bing & Redish 2009) and epistemic games (Tuminaro & Redish, 2004). These should provide a useful lens on the development of problem solving skills and permit an instructor to recognize the development of sophisticated problem solving behavior even when the student makes mathematical errors.

## 1. USING MATH IN PHYSICS: SUBTLER THAN IT LOOKS

Physics heavily uses math in an intimate way, but very differently from the way it is typically taught in a math class. We can illustrate this by analogy with a classic example often used in the education literature.

It is very important that you learn about traxoline. Traxoline is a new form of zionter. It is montilled in Ceristanna. The Ceristannians gristeriate large amounts of fevon and then bracter it to quasel traxoline. Traxoline may well be one of our most lukized snezlaus in the future because of our zionter lescelidge.

Answer the following questions in complete sentences. Be sure to use your best handwriting.

1. What is traxoline?
2. Where is traxoline montilled?
3. How is traxoline quaselled?
4. Why is it important to know about traxoline?

This example (attributed to Judith Lanier) is often used to deride standarized test problems that contain their own answers and do not require students to understand any of the science. But although this item does not test students' scientific knowledge, it definitely requires them to bring to bear considerable knowledge about English grammar, sentence structure, and the relationship of words.

In a very real sense, mathematics is like this example. It studies relationships among items that have undefined referents. We know no more than that they are numbers (or functions, or matrices, $\ldots$, etc.). When we do math in science, we expect our abstract relationships to be enriched by having our symbolic elements enriched by characteristics that have physical interpretations - units, transformation properties (e.g., scalar or vector), etc.

An example that illustrates how physics uses meaning to transform mathematics was given by [Dray & Manogue 2002]. Asked the question, "if $T(x, y) = k(x^2 + y^2)$ where $k$ is a constant, what is $T(r, \theta)$?" a mathematician might appropriately answer, "$T(r, \theta) = k(r^2 + \theta^2)$".

A physicist or engineer is far more likely to interpret the expression as representing some function at a point in physical space and interpret the variables as coordinates in the plane. She would then write the function (by what the mathematicians refer to as "an abuse of notation") as "$T(r, \theta) = kr^2$". This sort of reinterpretation of standard mathematical procedure using ancillary physical meaning is extremely common in physics.

The use of physical meanings play a number of important roles in the interpretation of math in science, including: helping to guide problem solution strategies, providing metacognitive warnings to facilitate error checking, and providing reasons to reject a particular mathematical model. In addition, the association of mathematics with ancillary physical information is what gives equations sense and meaning. [Redish & Smith 2009] This blending of physical and mathematical meaning is an essential part of three of the four steps in mathematical modeling process, as schematically displayed in the figure 1. [Redish 2005]

We are interested in how well physics majors learn to blend appropriate physical information into mathematical problem solving. To study this we have looked at their local functional epistemologies: when in the context of actually solving a problem, what kind of knowledge do

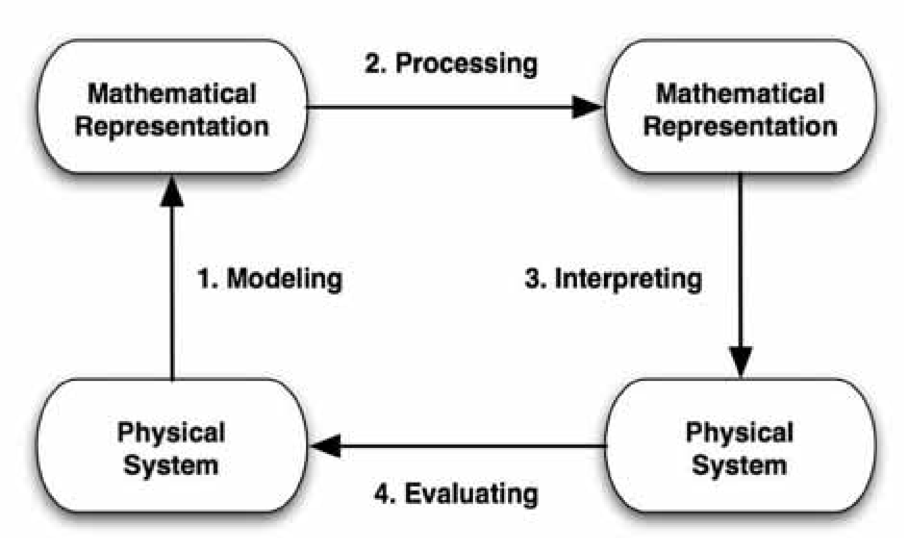


Figure 1: The mathematical modeling process.

they draw on? Equally interesting is the question, what kind of knowledge that might be useful or appropriate do they ignore or fail to deploy?

## 2. Problem Solving and Epistemology: How do students mix different kinds of reasoning?

In order to see how students mix mathematical and physical knowledge in learning how to solve complex mathematical problems, we have observed ~100 hours of students solving problems in groups in university physics classes from Introductory Physics to Quantum Mechanics.

What we learn is that novice and journeyman students do not have appear to have fixed epistemological stances, but rather they select from an array of possibilities. Their choice of what they think is needed to decide something is right can be labile, or it can "stick" - producing a narrowed selective attention that keeps them from accessing knowledge that they have learned and that would be useful.

## 3. The Ontology of Epistemology

In order to talk about how students use their knowledge in solving problems we need to have a language. We define three terms:

1. *Epistemological framing* - the judgement (often not a conscious one) as to what knowledge is relevant to the problem at hand;
2. *Epistemological resources* - broad general classes of warrants used to decide something is true;
3. *Warrants* - specific implementations of epistemological resources in the context of a particular problem or argument

## 4. WARRANTS GIVE EVIDENCE OF EPISTEMOLOGICAL FRAMING

Much of the reasoning we have observed in novice and journeyman physics students depends on one (or more) of four general types of proof. (Bing, 2008; Bing & Redish, 2008, Bing & Redish, 2009) These classes of warrants guide our analysis of their epistemological framing:

1. *Calculation* - algorithmically following a set of established computational steps should lead to a trustable result.
2. *Physical mapping* - a mathematical symbolic representation faithfully characterizes some feature of the physical or geometric system it is intended to represent.
3. *Invoking authority* - information that comes from an authoritative source can be trusted.
4. *Mathematical consistency* - mathematics and mathematical manipulations have a regularity and reliability and are consistent across different situations.

We can illustrate these with an example taken from a third-year physics majors' class in intermediate mathematical methods.

### CASE STUDY 1

In this example, three students are working together on a homework problem. This problem is meant to demonstrate to them the mechanism by which a conservative force leads to work being independent of the path and a well-defined potential energy. The students are not explicitly aware of this framing.

> Use the definition of work [$W_{A\to B} = \int_A^B F \cdot dr$ ] done by the force of gravity due to a mass $M$ on a small mass $m$ as it moves along each of the two paths from point A to point B as shown in fig. 2.

As a first step to solving this problem, the students have suppressed the common $GmM$ factor in the gravitational force law and written (incorrectly – they are forgetting the cosine factors from the dot product.)

$$\int_{\sqrt{2}}^{3\sqrt{2}} \frac{1}{r^2} dr = \left( \int_1^3 \frac{1}{y^2+9} dy + \int_1^3 \frac{1}{x^2+1} dx \right) \tag{1}$$

There is a long discussion of which the brief excerpt shown below is a part.

S1: No, No, No.
S2: They should be equal?
S1: They should be equal.
S2: Why should they be equal? This path is longer if you think about it.

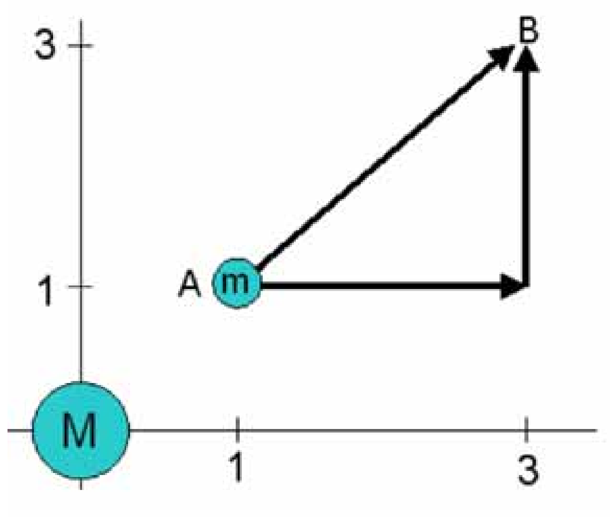


Figure 2: Case Study 1.

S1: Because force, er, because work is path independent.
S2: This path is longer, so it should have, this number should be bigger then.
S1: Work is path independent. If you go from point A to point B, doesn't matter how you get there, it should take the same amount of work.
S2: Ok. Well is this – what was the answer to this right here? What was that answer? ...Cause path two is longer than path one, so ...See, point six one eight, which is what I said, the work done here should be larger than the work done here cause the path ...

As the snippet begins, S1 is in a "by authority" frame. He knows the result (though he has written the equation incorrectly) and doesn't want to think about why. S2 is in a "physical mapping" frame. He believes the length of the paths should match with the numerical results. Both students retain their chosen frames for a number of minutes. Unable to convince S1, using physical reasoning, S2 turns to calculational warrants. Eventually, S1 begins to respond to S2's demand for a different class of warrant and constructs a conceptual physical example. Their negotiation of acceptable warrants leads them to find the error and agree on a result. [Bing & Redish, 2009]

## 5. CONCLUSION

Expertise in physics problem solving involves the ability to blend different epistemological framings and to flip quickly from one framing to another in response to snags and difficulties. The clustering of skills into frames can be an efficient way to proceed; if one can quickly recognize the tools needed to solve a problem, the problem can be solved without spending time hunting through a large number of possibilities. But when problems or inconsistencies arise, it can be more effective to explore a wider search space of possibilities. An instructor who is aware of the fact that students may “get stuck” in a framing that limits their access to tools and knowledge they may not only possess but be good at, will have a better understanding of the true nature of the difficulty the students may be experiencing.

## ACKNOWLEDGEMENTS

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# HOW TO PROMOTE THE LEARNING OF PHYSICS FROM FORMULAE?

Esther Bagno, Bat-Sheva Eylon & Hana Berger

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## ABSTRACT

We describe a study investigating students' "understanding performances" of basic formulae in physics. We found that students face difficulties in describing verbally the components of the formulae, in comprehending the correspondence between concepts and the relevant labels in the formulae, in specifying the conditions under which the formulae can be described and in manipulating the units of the formulae. Based on the findings, we developed cooperatively with high-school teachers, a generic classroom activity focused on the interpretation of a formula. The activity engages the student in active learning, through explication of prior knowledge, interaction with peers, use of multiple representations, relating formal learning to everyday experience and reflection. We report on a study investigating students' learning with this activity and its effects on teachers.

## 1. INTRODUCTION

Several recent reports highlight the importance of providing students with opportunities to form an integrated and coherent understanding of science at all levels of schooling (Roth, Druker, Garnbier, et al. 2006; Duschl, Schweigruber and Shouse, 2007; Linn, 2007). In learning about a certain topic in school, students bring their own ideas and in addition encounter a rich array of learning experiences. They need to form relationships among the pieces of knowledge that they acquire, otherwise, their knowledge will be fragmented and quickly forgotten (Perkins 1995).

Research has shown that many students do not sufficiently integrate their knowledge spontaneously and thus they need their teachers' assistance in this regard (Bagno & Eylon, 1997; Perkins, 1995). The relationship between mathematics and physics is one of the examples of this situation as evidenced, for example, in difficulties that students face with formulae.

In a diagnostic study (Bagno, Eylon & Berger, 2008) we found that students fail to relate between a formula and its physical meaning. They have difficulties to relate the formula's components and their units, to identify the formula's conditions of applicability, to interpret the physical meaning of its special cases, to describe the formula in multiple representations, to point out its role in problem solving, and to relate it to real-life experiences. These deficient performances constitute what Perkins & Blythe (1994) term "understanding performances" defined as the ability to do a variety of thought-demanding things with a topic.

Unfortunately, many physics teachers do not realize that their students' knowledge is fragmented. Those that do, usually organize the ideas for their students instead of adopting a learner-centered approach that allows the students to be active in the process of integrating their knowledge (Arons, 1997; Bagno & Eylon, 1997, Bagno, Eylon & Ganiel, 2000; Bell & Linn, 2000; Eylon, Berger & Bagno, 2007).

How can one support students in integrating their knowledge? Linn & Eylon (2006) have identified four interrelated processes that jointly lead to knowledge integration: *eliciting current ideas, adding new ideas, developing criteria for evaluating ideas and sorting out ideas.* They claim that in order to form integrated knowledge, students need opportunities to be engaged in these four processes and that the failure of many instructional programs to support knowledge integration is because one or more of these processes are missing. In this paper we describe an activity aimed to support students in understanding formulae. This activity provides opportunities for students to engage in the four knowledge integration processes described above. We report on a study investigating students' learning with this activity and its effects on teachers.

## 2. The activity: "Interpretation of a Formula"

The'Interpretation of a Formula" activity was developed jointly by experienced physics teachers and science education experts (see Figure 1).

The students fill in the worksheets of the activity engaging them in active learning, interaction with peers, using multiple representations, relating formal learning to everyday experience and reflection (Eylon & Reif, 1984; Chi, Feltovitch & Glaser, 1981; Bagno, Eylon & Ganiel, 2000; Reif & St. John, 1979). The activity takes 1-2 lessons to complete and consists of a five-phase cycle: (1) **Individual work**, in which the students, guided by a set of tasks, explicitly elicit their knowledge about the formula. (2) **Group work**, in which the students are working in small groups, on the same set of tasks, evaluate their individual work, add new ideas, and reach a consensus (or disagreement). (3) **Class discussion**, in which a representative of each group presents the group's consensus; all the issues raised in the group work are discussed, under the guidance of the teacher, and a classroom summary is formulated. (4) **Homework** on applications, in which the students use the formula in other formal learning experiences and real-life scenarios. (5) **Individual reflection** in which each student individually accounts for what he/she has learned in the previous four phases and identifies what still remains unclear. Since the activity is generic it can be used repeatedly with different formulae. It is assumed that by using the same activity several times, the students will form a "habit of mind" for using this activity on their own with new formulae leading to better knowledge integration between mathematics and physics.

## 3. The study

*Research questions*

1a What were the difficulties that students faced initially in relating the mathematical and physical aspects of formulae?

1b What progress did students make as a result of working with the activity?

1c What issues were raised by the students in their reflection on the activity?

**Individual work**

Consider the formula:

1. Write down using physics terms, the meaning in physics of each component of the formula (including units):

| Component | Meaning in physics | Units |
|---|---|---|
| | | |

2. Show that the units on the right side of the formula are identical to the units on its left side.
3. Under which conditions can the formula be applied?
4. Describe the relationship between the components of the formula either by a graph or by a drawing.
5. Using the following table analyze special / boundary cases for the formula (for example, one of the components is zero).

| The special case | The form of the formula in this case | The meaning of the formula in this case |
|---|---|---|
| | | |

6. If the formula contains assembled components (either multiplication or portion), write down their physical meaning (use the table below).

| The assembled component | The physical meaning of this component |
|---|---|
| | |

7. Write down, using your own words, the meaning of the formula.

**Group work**

Discuss with your group mates each of the questions in the individual work. If necessary, correct your work.

**Class discussion**

After the discussion in class, fill in again, the individual work sheet.

**Applications -Homework**

1. Does this formula make sense to you? Why (or why not)?
2. Describe an everyday scenario or a physics problem in which the formula applies.
3. Describe a special case of the above scenario or the physics problem
4. What change do you have to make in the everyday scenario or problem that you described in 3, so that the formula will not apply anymore?

**Individual reflection**

1. What did you learn from the activity?
2. What did the group discussion add to your understanding?
3. What did the class discussion add to your understanding?
4. What is still unclear to you?

Figure 1: The ‘interpretation of a formula’ activity.

2a How did the activity influence the teachers' practice?

2b What issues were raised by the teachers in their reflection on the activity?

*Method*

A group of nine teachers experienced the activity as students and then implemented the activity in their classes on eight different formulae with 260 students.

*Research tools and analysis*

All the worksheets of the students were collected and analyzed. The research tools aimed at:

1. Diagnosing the students' initial state of knowledge (based on the individual work phase of the worksheets)
2. Evaluating the students' progression in knowledge (based on the transition throughout the activity from individual, to group, to all-class phases)
3. Examining the students' reflections on the activity.

Informal interviews with all the teachers were conducted during and after the instruction. We also observed some of the teachers' classes.

*Results*

**1a. What were the difficulties that students faced initially in relating the mathematical and physics aspects of formulae?**

The students faced difficulties in the above mentioned "understanding performances" of the formulae. In figure 2 we present four examples of these difficulties concerned with manipulation of units, conditions of application, verbal representation and special cases.

**1b. What progress did students make as a result of working with the activity?**

We have found that as a result of the activity, students progressed in relating the mathematical and physical aspects of a formula. For example, figure 3 presents the decrease in the number of students expressing only mathematical and not physical conditions for applying a formula.

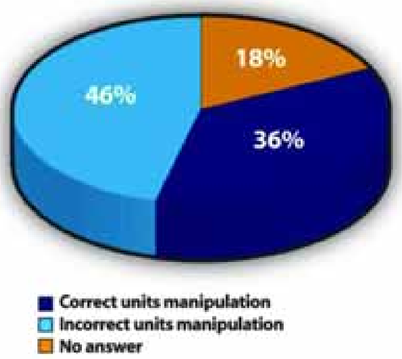
**1c. What issues were raised by the students in their reflection on the activity?**

Students claimed that the activity contributed to their ability to relate the mathematical and physical aspects of a formula. For example, a student wrote in the reflection phase "*It helped me understand a formula logically and not only through using the formula in calculations*" and another student wrote "*I have learned the physical meaning of the assembled components of the formula*". In sum, 57% of the students claimed that the activity supported them in the use of physics in explaining the formulas.

**Manipulating units**

*"Show that the units on the right side of the formula are identical to those on the left side"*

**260 students, 8 formulae**




Example of incorrect manipulation of units for

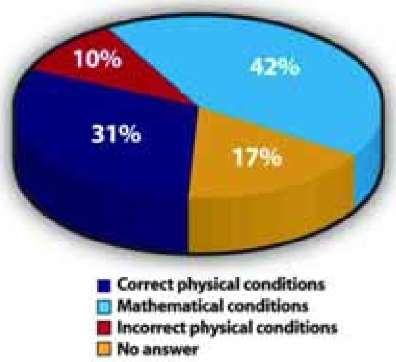
$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$[Meter] = [Meter] + \frac{[Meter]}{2}$$

**Conditions of Application**

*"Under which conditions can the formula be applied?"*

**103 students, 3 formulae**




Example of mathematical condition for

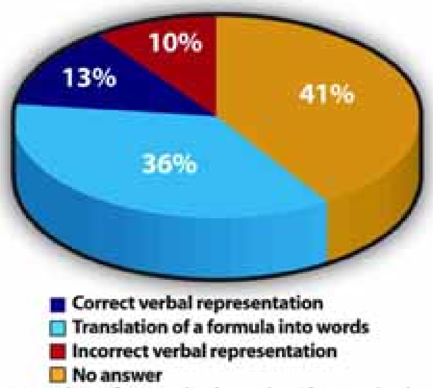
$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

*"We may use the formula whenever we have to calculate an unknown quantity and the other quantities are given"*

**Verbal Representation**

*"Write down, using your own words, the meaning of the formula"*

**72 students, 3 formulae**




Example of translating the formula into words for

$$\Sigma \vec{F} = m\vec{a}$$

*"The net force equals to the mass multiplied by the acceleration"*

**Special Cases**

*"Analyze special/boundary cases of the formula"*

**260 students, 8 formulae**


For example, a convex lens was not considered as a special case of the formula

$$\frac{1}{v} + \frac{1}{u} = \frac{1}{f}$$

By the majority of the students in one of the classes

Figure 2: Students' difficulties in relating the mathematical and physics aspects of formulae as appeared in their individual worksheets.

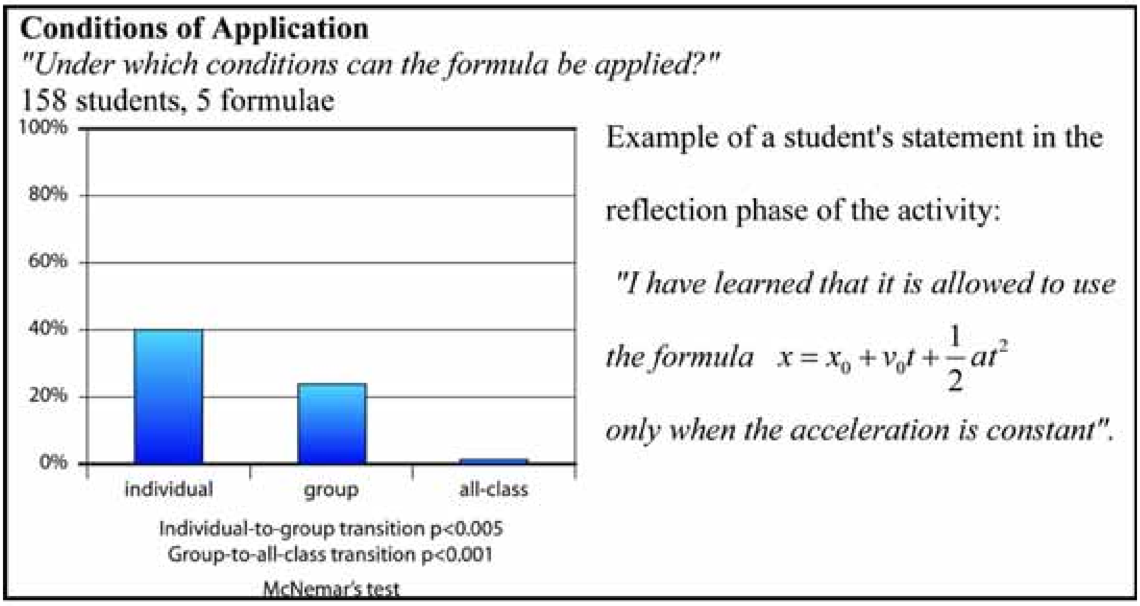



Figure 3: The percentage of students who expressed only mathematical conditions of applying a formula while progressing along the phases of the activity (N = 158).

**2a. How did the activity influence the teachers' practice?**

As a result of working with the activity

- Teachers discussed formulae with colleagues and enriched their knowledge. For example, following the analysis of students' worksheets regarding the formula $\Sigma F = ma$, teachers debated whether "The first law of Newton is a special case of the second law of Newton or not".
- Teachers used the activity as a diagnostic tool. For example, one teacher reported: "During the examination of my students' worksheets I was astonished to find students' learning difficulties that I had not been aware of beforehand. Another teacher gave the following example, "the potential energy $U$ at any distance from the earth's center $r(r > r_E)$, with the choice for $U = 0$ at infinity, is positive".
- Teachers used the activity to relate the mathematical and physical aspects of a formula in the all-class discussions as can be seen in figure 4.

**2b. What issues were raised by the teachers in their reflection on the activity?**

Teachers stated that the activity promotes students' understanding of formulae. In particular, it serves as a systematic tool for analyzing formulae: For example, "*This activity guides the students in analyzing a formula in a systematic way: to explain the meaning of its components, to check the units, to find the special cases and so on*" and "*On the one hand, the activity helps the students to take a whole formula, and decompose it into its components. On the other hand, it helps the students understand the connections among these components*".

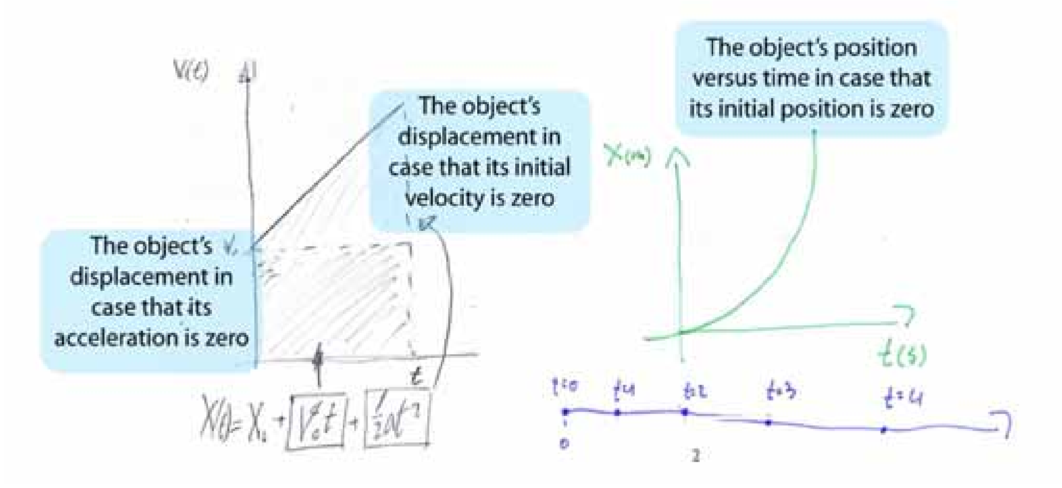



Figure 4: Graphs drawn by Physics teachers.

## 4. Summary and Discussion

This study indicates that beyond serving as a diagnostic tool for students and for teachers, the "Interpretation of a Formula" activity provides learners with a systematic method for understanding a formula. In particular, the activity enables learners to relate mathematical and physical aspects of a formula.

The activity promotes active learning by eliciting prior knowledge, discussing it with peers and modifying it accordingly, as claimed by Arons: "*Students must have time to form concepts, think, reason, and perceive relationships. They must discuss ideas and they must write about them*" (Teaching Introductory Physics. p. 364).

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# RECOGNIZING THE STRUCTURAL ROLE OF MATHEMATICS IN PHYSICS LESSONS

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## ABSTRACT

Physics and Mathematics have been deeply interrelated since the very beginning of scientific knowledge and this mutual influence has played an essential role in both their developments. Historical case studies show us both physical problems motivating the creation of mathematical concepts as well as mathematics previously originated in the "abstract world" being used and physically interpreted. However, in the context of education, these two disciplines tend to be treated separately and the students hardly become aware of this successful interplay. In this work, we propose a categorization that aims at distinguishing *technical skills* - the ones related to the domain of basic rules of mathematics and normally developed in maths classes from the *structural skills* - which are related to the capacity of employing the mathematical knowledge for structuring physical situations and recognizing their interrelations. Our main goal is to establish cognitive abilities to be developed in physics education with the purpose of acquiring the latter ones.

## 1. INTRODUCTION

A brief look at the history of Mathematics allows us to realize that several mathematical concepts have their origin in genuine physical problems. Just to mention a few examples: Einstein (1921) considered geometry to be one of the oldest physical theories; the origin of calculus is practically inseparable from the description of movement so that in Newton's theory of fluxions, the fluent variable was the time and the fluxion the instantaneous velocity (Boyer, 1949); according to Poincaré (1970), differential equations were originated in the core of physics to solve physics problems; vector algebra and analysis is directly related to the mathematization of electromagnetism (Silva, 2007); Fourier analysis was motivated by problems of waves on strings and propagation of heat (Davis & Hersh, 1981), amongst many others.

In an inverted reasoning, mathematical concepts created in an "abstract world", without any compromise with applications to the "real world", are commonly "used" by physicists to construct their theoretical explanations of phenomena. Some philosophers use the analogy of *prefabricated mathematics* and compare the physicist's attitude to "a person who goes to the market of mathematics to take what he needs to construct his theory" (Boniolo & Budinich, 2005, p. 83). This is actually the case for the conic sections, initially studied by Apollonius of Perga in the III century B.C. and used almost two thousand years later by Kepler to describe the movement of the planets. Another interesting episode is the rising and development of complex numbers in the XVI century, motivated by the mathematical problem of finding solutions for cubic equations, and two hundred years later, these "imaginary" numbers were being used and physically interpreted initially in the optics of Fresnel, and afterwards in electrodynamics and quantum mechanics (Bochner, 1981). More recently, other examples of prefabricated mathematics are found in the application of noneuclidean geometry and tensorial calculus in general relativity and the use of Hilbert space in quantum mechanics.

These remarkable historical relations between mathematics and physics can (and should) be made explicit in the context of science education. Unfortunately, what we find in regular classrooms in Brazil is far away from that goal. Recent studies on Brazilian students' scientific conceptions (Karam, 2007; Ricardo & Freire, 2007) show us that they hardly recognize the structural role of mathematics in physical thought and don't realize the importance of physics for the development of mathematical concepts. Our main verifications are that: i) Students lack understanding of the conceptual meaning of physics equations; ii) In physics classes, mathematics is normally regarded as a mere tool to solve problems; iii) In maths classes, physics tends to be only an application of previously defined mathematical abstract concepts; iv) Many mathematical concepts are learned without any relation to the physical problems that originated them; v) No integrated curricula is implemented or even discussed; and vi) Students memorize equations instead of deriving them from physical principles.

From what has been seen in the international literature, it is possible to infer that similar problems are found worldwide. Defending the importance of modeling for physics education, Angell and others (2008) argue that the understanding of the physical description of phenomena involves the ability to represent these phenomena through multiple representations and being able to translate between them. One of the conclusions of their research is that translating from physical situations to the formalized language of mathematics is the most difficult task for students. A similar result was obtained by Tuminaro & Redish (2007) when they analyzed and categorized students' use of mathematics when solving physics problems. The theoretical framework proposed by the authors consists of six hierarchical structures (Epistemic Games), which allow them to understand students' reasoning when solving physics problems. The most intellectually complex e-game is called Mapping Meaning to Mathematics and it involves the ability to translate the conceptual physical story into mathematical entities and relate them to the same story. This was recognizably the most difficult task for the students in the research.

Aiming at facing up to this challenge, we propose a distinction between *technical* and *structural skills* when it comes to analyzing the students' use of mathematics in physics and the comprehension of their interdependence. In this paper, we describe a set of five structural skills and argue that they should be aimed at in physics (and mathematics) education. Taking into account that mathematics has a structural role in physical thought (Pietrocola, 2008), these skills were inspired both by epistemological and empirical studies.

## 2. Technical versus structural skills

One of the main concerns of physics teachers, whichever level of education, is their students' knowledge of mathematics. It is rather common to find teachers complaining that their pupils don't know enough math and, therefore, aren't able to succeed in physics.

In fact, it has already been sufficiently demonstrated that the absence of some basic mathematical skills is a considerable factor for students' failure in physics courses (Hudson & McIntire, 1977; Hudson & Liberman, 1982). However, it is equally consensual that the domain of these skills doesn't guarantee success in physics; in mathematical terms: they are necessary but not sufficient. Pre-course tests of algebraic and trigonometric skills taken by approximately 200 students initiating a physics course allowed Hudson & McIntire (1977) to conclude that the mathematical pre-test was more a "predictor of failure than a guarantee of success" (p. 470).

This lack of correlation between the domain of mathematical skills and success in physics courses can be understood if we recognize that using mathematics in physics is something different from simply doing math. This is exactly the position defended by Redish (2005) when he states that:

> [. . . ] using math in science (and particularly in physics) is not just doing math. It has a different purpose - representing meaning about physical systems rather than expressing abstract relationships - and it even has a distinct semiotics - the way meaning is put into symbols - from pure mathematics. It almost seems that the "language" of mathematics we use in physics is not the same as the one taught by mathematicians (p. 1).

Some of these differences are pointed out by the author, such as:

> - We [physicists] have many different kinds of constants - numbers $(2, e, \pi, \ldots)$, universal dimensioned constants $(e, h, k_B, \ldots)$, problem parameters $(m, R, \ldots)$, and initial conditions.
> - We blur the distinction between constants and variables.
> - We use symbols to stand for ideas rather than quantities.
> - We mix "things of physics" and "things of math" when we interpret equations.
>
> But perhaps the most dramatic difference is the way we put meaning to our symbols (Redish, 2005, p. 2).

In agreement with Redish (2005), we believe that it is interesting to propose a distinction between technical and structural skills when it comes to analyzing the student's use of mathematics in physics and the comprehension of their interdependence.

The first ones - *technical skills* - are normally developed in maths classes and are related to the technical domain of mathematical systems, such as operations with algorithms, solution of equations, etc. Many physics teachers associate their students' failure with the lack of these technical skills and it is fairly common to encounter physics teachers complaining that the students cannot "divide with fractional numbers, isolate a variable, solve an equation, calculate the value of a determinant and so on . . . " It is not uncommon for these teachers to struggle in the physical interpretation of problems, even presenting the function that represents the problem's solution, and then say: "from now on it is only mathematics and the solution to this was already presented to you in a previous subject or in maths class". According to Pietrocola (2008), this implies that once the problem has been understood, from a physical point of view, from then on such competencies are no longer that teacher's responsibility. The transformation of the problem is a mathematical algorithm and solving this would depend on skills learned in other subjects.

This posture reflects a naïve or incorrect conception of the deep interrelations between mathematics and physics, once it implies that the former is considered to be a mere tool for the latter (Pietrocola, 2008). When analyzing some standard physics problems, such as the ones usually found in textbooks, it is not uncommon to realize that the necessary skill to solve it is to find the correct formula, "plug" numbers into it and reach the numerical solution. Those kinds of problems and exercises focus mainly on the "hows", instead of on the "whys".

Accordingly, beyond the technical skills, we propose a set of cognitive abilities - namely *structural skills* - which are associated with the capacity of employing mathematical knowledge for structuring physical situations and recognizing their mutual influence. In this sense, Pietrocola (2002) defends that:

> Considering that mathematics is the language that allows the scientist to structure his/her thought in order to comprehend the physical world, science teaching should enable the students to acquire this ability. [...] it is not about just knowing Mathematics and applying this knowledge to physical situations, but being able to apprehend theoretically the "real" through a mathematical structure (Pietrocola, 2002, pp.110-111).

Hence, we propose that the acquisition of structural skills should be one of the main goals of physics education. From both epistemological and empirical studies, we have reached a set of five of these skills (competences), which are presented and discussed.

### 2.1. DERIVE EQUATIONS FROM PHYSICAL PRINCIPLES

Undoubtedly, one of the most important notions of mathematical reasoning is the concept of proof. This notion, which was mainly developed by the Greek mathematicians and philosophers, involves starting from an "evident" set of postulates and axioms and, by logical deductions, being able to prove a certain theorem. This style of reasoning is widely used and exemplified in Euclid's Elements and can also be found in several Physics masterpieces, such as Newton's Principia and Einstein's paper on Special Relativity. In spite of the controversial philosophical debate around the "veracity" of the axioms, the idea of proving is more deeply related to the capacity of answering why questions. Presenting and discussing some proofs of the Pythagorean Theorem to the students for example, is definitely very different (better) than simply stating it as a mysterious truth and using it in several exercises.

In physics education, it is also possible to highlight the reasons by demonstrating physical formulae. In this sense, we believe that derivations enhance student's knowledge about the origin of physics equations, allow them to penetrate into the inner structure of physics reasoning and avoid the rote memorization of senseless mathematical formulas. Some good examples are the derivation of the law of refraction from Huygens' or Fermat's principle or Kepler's laws from Newton's dynamic laws. In fact, different ways of deriving the same formula is also recommendable and provide exciting discussions, since it clarifies the flexibility of mathematics.

An interesting approach, which encourages fruitful debates, is confronting these mathematical derivations with the process of obtaining the same law from the analysis of experimental data.

### 2.2. Identify the essential aspects that justify the use of mathematical structures in physical phenomena

Students should be aware of the reasons why a certain mathematical structure is useful for describing a particular physical phenomenon. For instance: trigonometric functions are valuable in physics either because it is necessary to obtain orthogonal components of a vector or because there is some periodicity related to the phenomenon (waves, electric circuits, etc.). Matrices (vectors and tensors) are necessary when the various states of a physical system are represented by groupings of real numbers which cannot be isolated interpreted, each by itself. Returning to the analogy of prefabricated mathematics, it would be as if one were able to find the aisle in the "market of mathematics" where he or she could find what is needed to model a particular phenomenon.

This skill could be accessed by encouraging students to think about the reasons why each physical formula has that particular shape. Some examples can be: What is $\pi$ doing on the simple pendulum formula? Why is the work done by force a scalar product and the torque a vector product? Why is there a logarithm in Boltzmann law? Naturally, this ability can also be developed in the context of mathematics education if the (physical) motivations for the creation of certain mathematical concepts are mentioned to the students.

### 2.3. Understand the conceptual meaning of physical equations

What does a physical formula mean for the pupils? Tuminaro and Redish (2007) have showed that many students blindly plug quantities into physics' equations and churn out numeric answers without conceptually understanding the physical implications of their calculations. In their framework, the authors identify this reasoning as an Epistemic Game called "Plug-and-chug". One emblematic example found in their results is when a student tries to use $PV = nRT$ to solve a problem and identifies the $R$ as being a radius, which has absolutely no relation to the physical context of the problem.

Therefore, it is imperative that students develop the ability of "reading" physical equations and interpreting their meaning. The learning tool described and analyzed in the contribution from Bagno, Eylon and Berger seems to be extremely useful for developing this skill.

### 2.4. Recognize the importance of analogical reasoning

Noticeably, one of the most fruitful resources of reasoning in physics is analogy, since the relation between the model and the modeled phenomenon is generally analogical. According to Mary Hesse (1953, pp. 202), "an analogy in physics is a relation, either between two hypotheses, or between a hypothesis and certain experimental results, in which certain aspects of both relata can be *described by the same mathematical formalism*". This last aspect is particularly important for the development of new theories by formal analogy. In his Analytical Theory of Heat, Fourier (1878, pp. 8) has highlighted the power of analogy by arguing that "mathematical