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Problem Solving and the Use of Math in Physics Courses

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PER: Using the methods of science to learn about science teaching & learning

- Physics Education Research (PER)
  - using the tools and methods of science
    to study and improve
    the teaching and learning of physics.
- An interdisciplinary effort involving
  - Physics
  - Education research
  - many other areas of study

The UMd PERG:

- **Faculty**
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  - David Hammer (Phys/C&I)
- **Research Faculty**
  - Andy Elby (Phys)
  - Rachel Scherr (Phys)
- **Postdoc**
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- **Grad Students**
  - Luke Conlin (C&I)
  - Adam Berman (Phys)
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  - Colleen Gillespie (C&I)
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  - Eric Kuo (Phys)
  - Mike Hull (Phys)
  - Mattie Lau (C&I)
  - Tim McCaskey (Phys)
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  - [Tom Bing (Phys)]
  - [Jonathan Tuminaro (Phys)]

Physics
Curriculum & Instruction
Both

[…] = alumnus

What are we trying to do when we teach physics?

- When we teach physics to a variety of populations, we hope they will do more than memorize a few definitions, equations, and facts.
- We are trying to teach a way of thinking about the physical world.
- A critical component is to learn to think about new situations using the general principles they have learned.
- A primary tool for teaching this is **problem solving** – a place where they have to think, not just parrot back answers.

What have we learned in PER?

- Constructivism –
  - Students build new knowledge by interpreting new information in terms of what they know.
- Misconceptions –
  - What students bring in to a physics class can lead them to misinterpret what they are supposed to be learning.
- Active learning –
  - Traditional “passive” environments are not as effective as research-based “active engagement” environments.

What about problem solving?

- Novice problems solvers in physics differ from experts in many ways.
  - Novices have less knowledge.
  - Novices knowledge is poorly organized compared to experts.
  - Novices tend to classify problems incorrectly, activating the wrong knowledge and tools.

Hsu, Brewe, Foster, & Harper, Am. J. Phys. ***72*** *1147 (2004)*
Maloney, in Handbook of Res. on Sci. Teaching & Learning (Macmillan, 1993).

Theory

- A classroom is a highly complex system. There are many components to talk about.
  - The structure and function of the learner
  - The structure and function of the classroom environment (on many levels)
  - The interaction of the learner with those environments.
  - The structure of the knowledge to be learned
    - through the perspective of the structures of the learner and learning environments.

Theory

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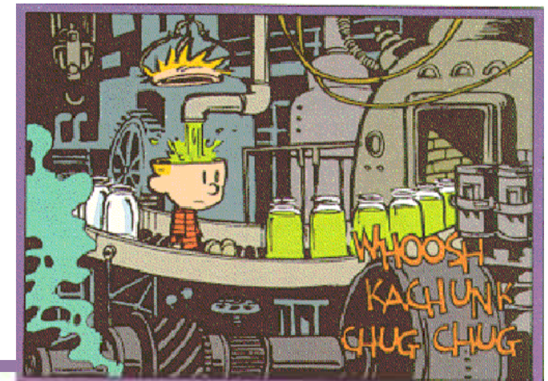
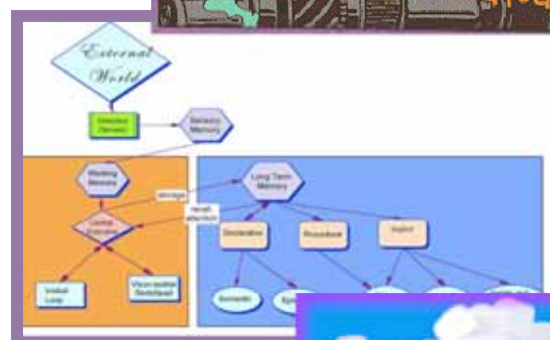
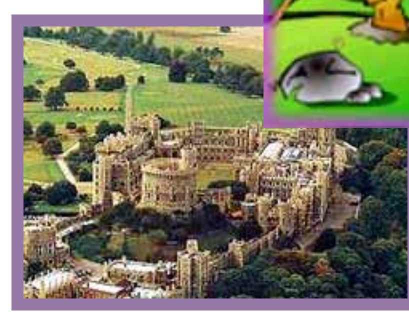
Modeling the Student

- To understand what we can (and can't) do, it helps to understand how the system we are trying to modify (the student) "works."
- In the past 50 years, much has been learned about thinking and behavior—but there is a lot of dross.
- How can we separate the gold from the slag?







The Resource Framework: Four foothold ideas

1. Activation / Resources
   - A perception / awareness ("cognit"*) of something corresponds to the activation of a set of linked neurons.
2. Association
   - The activation of one cognit can lead to the activation of others ("spreading activation")
3. Binding
   - Different cognits can become tightly tied so they always activate together – the user becomes unaware of their separate parts.
4. Selective attention / Control
   - Contexts can suppress, prime, or activate clusters of cognits.

Hammer, Am. J. Phys. Suppl. ***68*** *S52-S59 (2000)*
Redish, Fermi Summer School Lectures (2003)
**Fuster, Memory in the Cerebral Cortex (MIT Press, 1999).*

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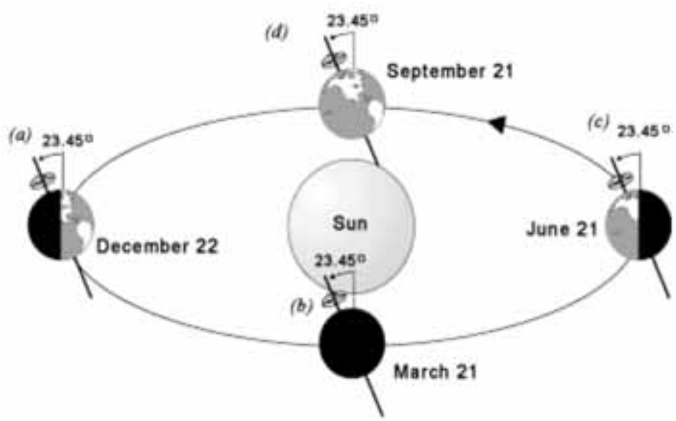
1. Activation: Resources

- Thinking is dynamic.
- Different knowledge elements or processes (resources) “turn on” and activate other related elements.
- Which related elements are turned on depend on context
- Things that “pop up” can become tightly tied to other elements.

Example:
Why do we have seasons?



- Essentially every elementary school student in the USA has been given the explanation.
- Then why do Harvard graduates give the wrong answer when asked?

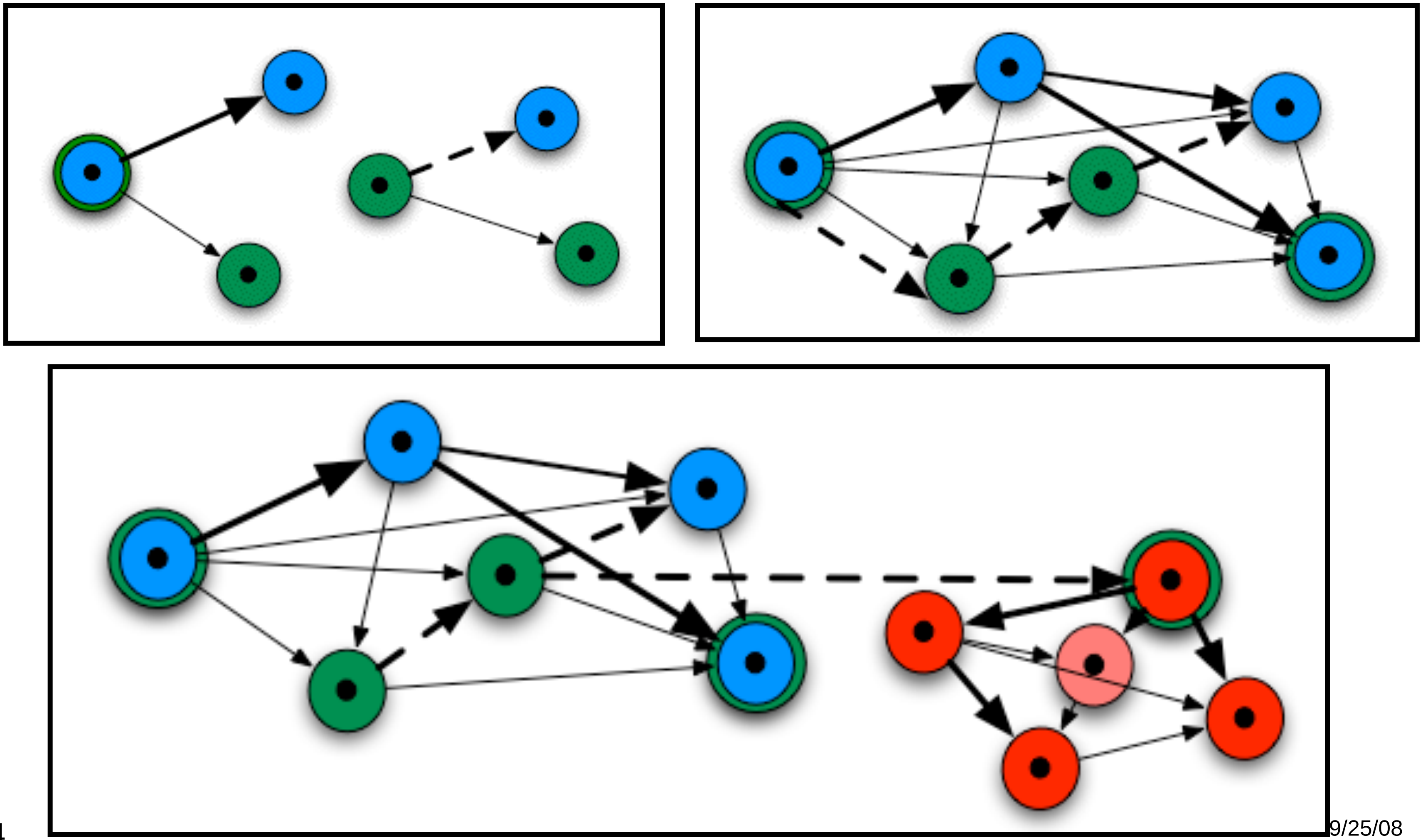
Primitive: Closer is stronger / more effective (neither right nor wrong)

Facet: You can get warmer by standing closer to the fire.(right)

Facet: It's warmer in the summer, so we must be closer to the fire.(wrong)

A Private Universe, http://www.learner.org/resources/series28.html

2. Association



Sabella and Redish, Am. J. Phys., ***75***, *1017 (2007)*

Memorize these numbers

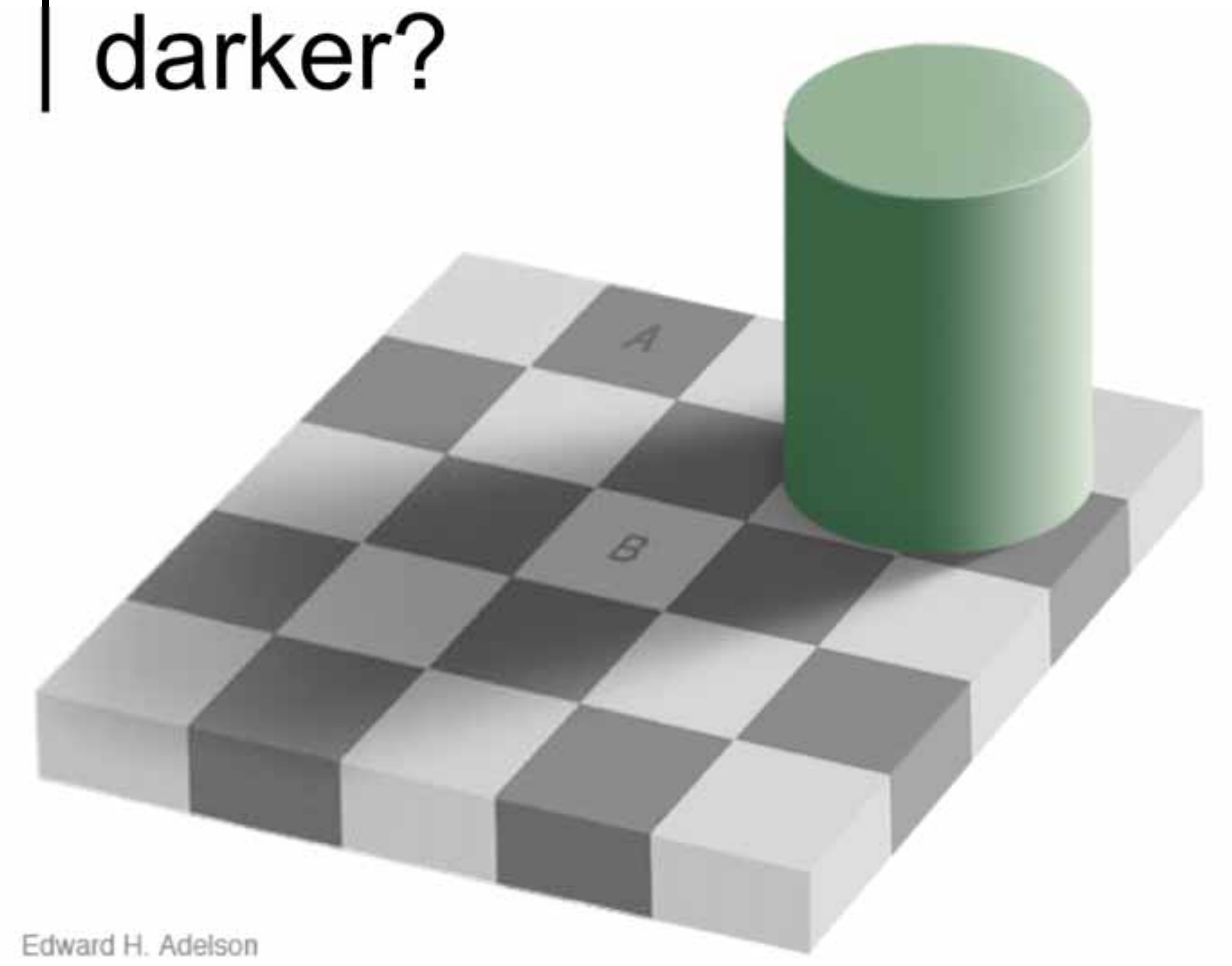
3 5 2 9 7 4 3 1 0 4 8 5

1 4 9 2 1 7 7 6 2 0 0 8

3. Binding

- As we learn, we bring together many different pieces of knowledge, binding them into a single coherent unit.
- Sometimes this process is very fast, sometimes it takes seconds, sometimes it takes years.
- “Compilation”

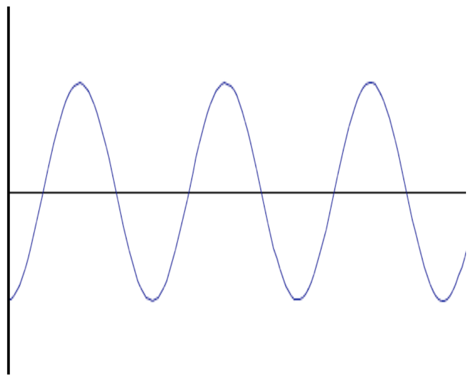
Which square is darker?



http://web.mit.edu/persci/people/adelson/

Binding is hard to undo!

- Our processing of visual signals is highly relative and impossible to “unpack”.
- Things we learn can also compile — sometimes over a period of years.
- Can you look at a graph and <u>not</u> immediately know where the derivative is 0?

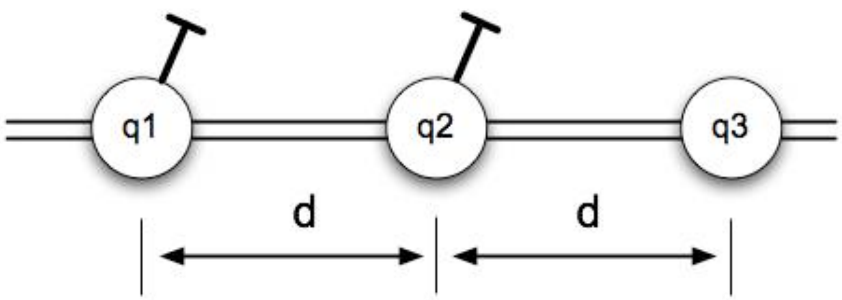


The Data

- *Learning How to Learn Science (2000-2004)*
  - 4-year NSF supported project
    to study algebra-based physics
  - All parts of the course were modified to
    - increase active engagement
    - focus on epistemological development
    - provide observational data ("ecological")
  - Approximately 1000 hours of videotaped data were collected in lab, tutorial, and HW center.
- *Learning the Language of Science (2005-2009)*
  - 4-year NSF supported project to study
    use of math in upper division physics
  - So far, about 50 hours of videotapes have been collected of students working on HW.

Is this a "simple" problem?

- Three charged particles lie on a straight line and are separated by distances d. q_1 and q_2 are held fixed. q_3 is free to move but is in equilibrium (no net electrostatic force acts on it). If q_2 = Q, what value must q_1 have?



- Circumstance:
  - Four students working in the course center.

Redish, Scherr, & Tuminaro, Phys. Teach., ***44****, 293 (2006).*

An hour?

- When we first viewed the video
  we were concerned that they took
  so long to solve what (on the surface)
  seemed to be
  a relatively simple problem.
- After a careful analysis,
  we became convinced that
  the work they did was worthwhile
  and a valuable part of their learning.

How they get there

Description of events
They make some progress thinking qualitatively, but are at first unsure about forces, directions, and fields.
The Teaching Assistant suggests they draw a diagram so they can agree on what is happening.
They now agree on which charges are exerting which forces in which directions and settle on a factor of -2.
One student, recalling a result of the non-linearity in a previous problem tries to get them to think using the equation (Coulomb's law).
Eventually, she manages to turn their attention to using the equation and she works out the correct solution to the problem using algebra — constructing a clean proof. The group is convinced.

Why so long? The professor's "simple" solution involves lots of hidden resources. Our list:

- Like charges repel, unlike attract
- Attractions and repulsions are forces
- Forces can add and cancel (one does not "win"; one is not "blocked")
- "Equilibrium" corresponds to balanced, opposing forces (not a single strong "holding" force)
- Electric force both increases with charge and decreases with distance from charge
- Objects respond to the forces they feel (not those they exert)
- Charges may be of indeterminate sign and still exert balancing forces on the test charge
- "Fixed" objects don't give visible indication of forces acting on them; "free" ones do
- Only forces on the test charge require analysis
- Each other charge exerts one force on test charge
- Each force may be represented by a vector
- "Equilibrium" corresponds to opposing vectors
- Vertical and horizontal dimensions are separable
- One dimension is sufficient for analysis
- Electric force both increases with charge and decreases with distance from charge
- Electric force decreases with the square of the distance

In this case, the students do what I want them to.

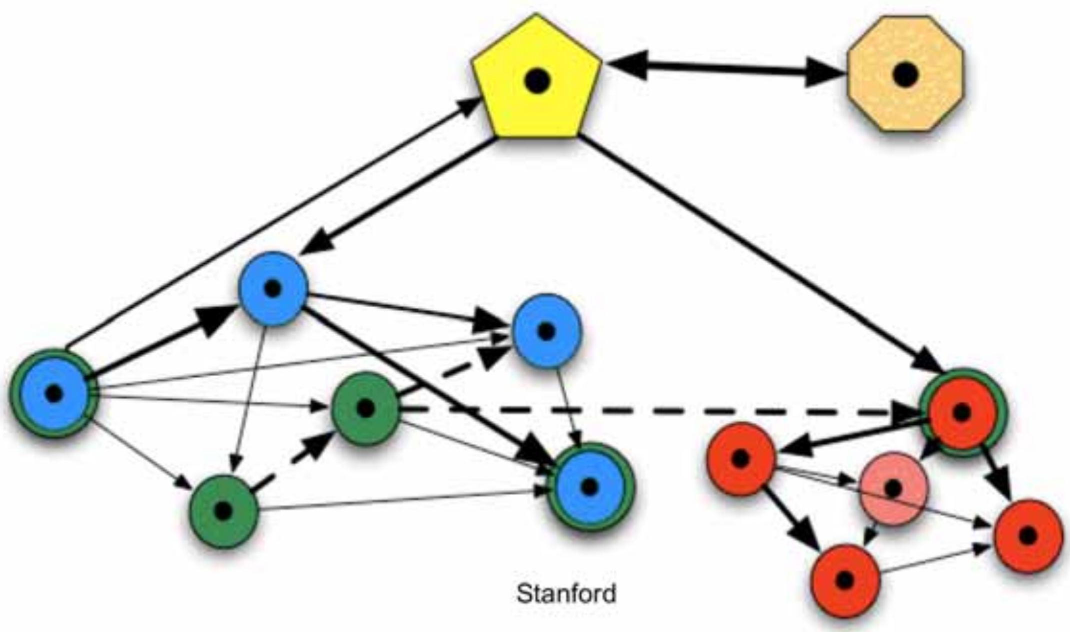
- They first make qualitative sense of the problem.
- Then they:
  - – nail down what they remember from their study of Newton's laws
  - – clarify the nature of the electric force
  - – estimate a qualitative result
  - – refine it by applying the quantitative principle – Coulomb's Law (correctly).

Reverse engineering expert knowledge

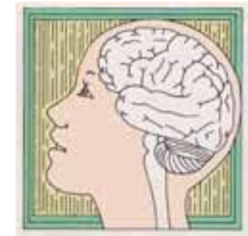
- I had failed to appreciate how much was compiled into my “simple” solution.
- Watching these students helped me “reverse engineer” what I had built over many years into a tight, automatic knowledge structure.
- The students are not only solving a problem. They are compiling the knowledge required for the problem and are learning how to solve problems in general.
- The fact that they are willing to work for an hour on a “short” problem is notable.

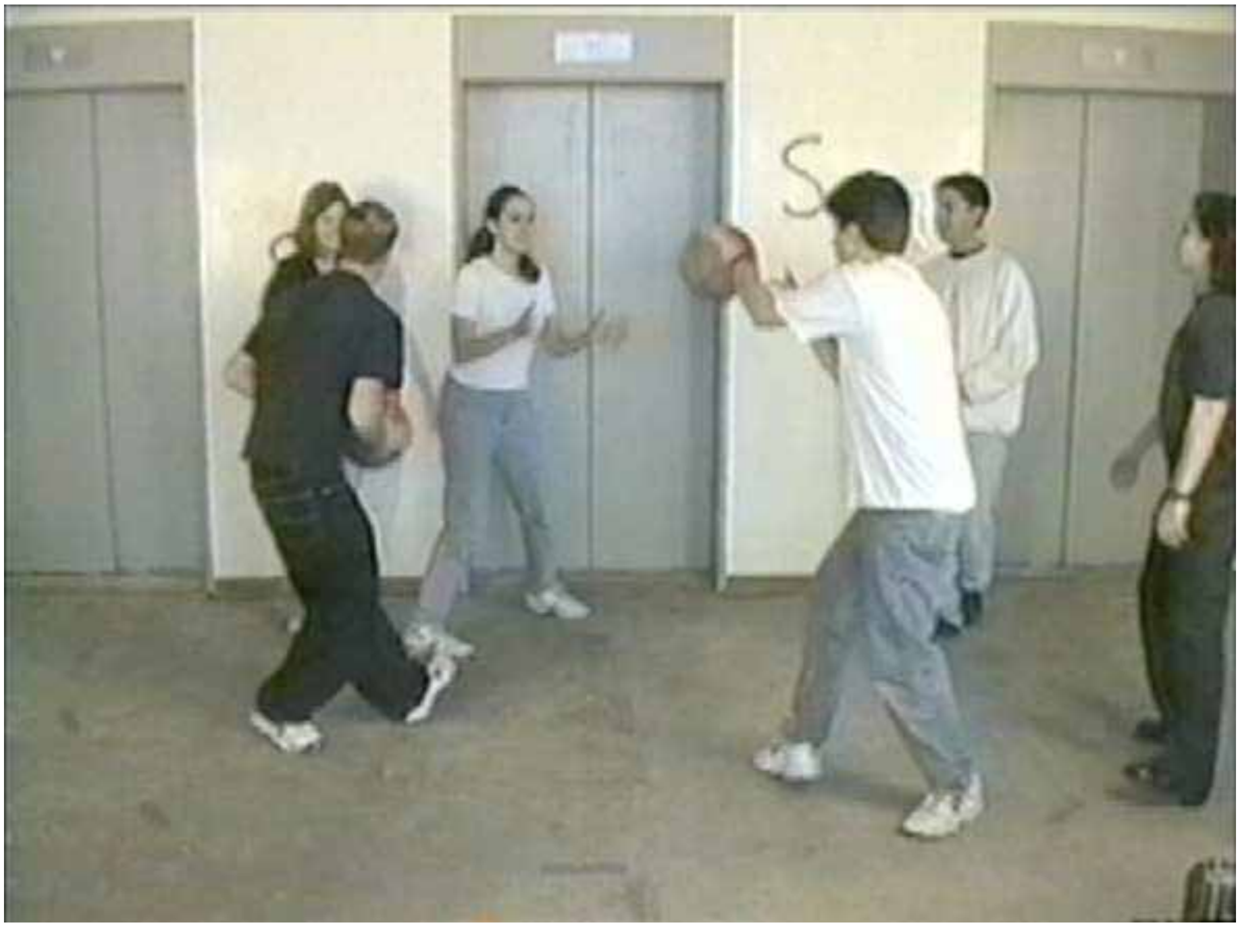
4. Selective Attention/ Control

- Synapses can be excitatory or inhibitory.
- The brain is filled with both feedforward links (for association and activation) and feedback links (for switching and context dependence).



Count the passes!





Selective Attention / Expectations

- One way control plays out is through selective attention – what we expect (often tacitly) is going on and relevant.
- There is too much in the world for our brains to process at once.
- We learn to select and ignore, *framing* our situation — deciding what matters and what doesn’t quickly and (often) unconsciously.

I. Goffmann, Frame Analysis (1997)

D. Tannen, Framing in Discourse (1993).

Framing a situation: What's going on here?

- A Selective Filter
  - An individual's expectations activate
    what to pay attention to and what to ignores
    in response to the 10,000 things and their interactions.
- Experience
  - College students have had many years of schooling and think they know what to expect in class.
- Expectations
  - If the students' expectations about what to do fail
    to match the teacher's, both may be disappointed.

Framing problem solving: The role of epistemology

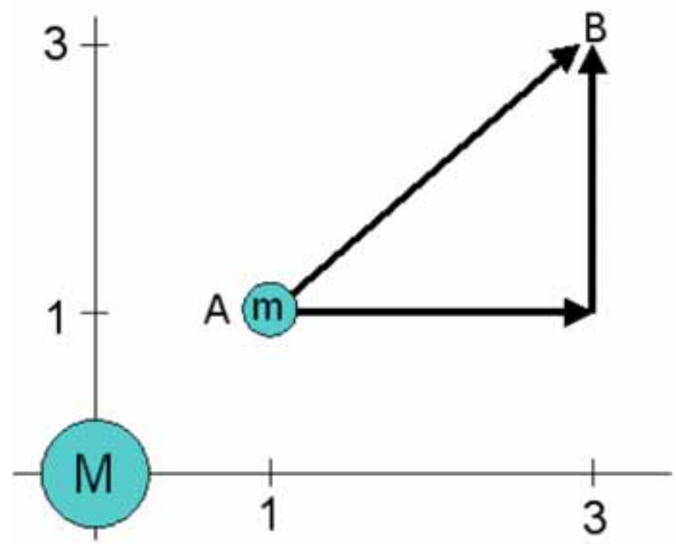
- When faced with a physics problem, “what’s going on here” involves a lot of epistemology.
  - “What of all the things I know is relevant here?”
  - “What of all the things I know are appropriate to use here?”
- *Note: These questions might not have the same answer!*

T. Bing, PhD Dissertation, U. of Maryland (2008).
http://www.physics.umd.edu/perg/dissertations/Bing/

The data:
Group work on problem solving

- A good place to observe student thinking is when they are discussing solving a problem with each other.
- It is particularly interesting to look at the kind of arguments (warrants) they use to support their claims when someone disagrees with them.
- We observes ~50 hours of students working on HW in upper division physics classes.

The problem



- A rocket (mass m) is taken from a point A near an asteroid (mass M) to another point B. We will consider two (unrealistic) paths as shown in the figure. Calculate the work done by the asteroid on the rocket along each path. Use the full form of Newton's Universal Law of Gravitation (not the flat earth approximation "mg"). Calculate the work done by using the fundamental definition of work:

$$W_{A\to B} = \int_A^B \vec{F} \cdot d\vec{r}$$

A proposed solution

- Three students worked on this problem in my course in Intermediate Mathematical Methods.
- They successfully wrote down and did the integral along the direct path
- S1 then wrote integrals for the value of the work along the indirect path.

$$\int_{1}^{3\sqrt{2}} \frac{dr}{r^2}$$

$$\int_{1}^{3} \frac{dx}{x^2+1} + \int_{1}^{3} \frac{dy}{9+y^2}$$

An argument ensues

- S2 insists that the integral along the indirect path must be longer because the path is longer.
- S1 refutes that by saying they have to be the same because "it's a definition."
- The argument continues for ~10 minutes

Elements of the argument

S1

- You should get a different answer.
- Why should they be equal? This path is longer.
- Just sum those up. I want the whole total.
- See, the work should be larger.

S2

- No, no, no, no, no, no, no.
- Work is path independent by definition.
- If you are at the bottom of a hill and want to drive to the top…it takes the same amount of energy to get from the bottom to the top [whichever path you take].

Classifying warrants: Epistemic modes

- Invoking authority
  - Remember a result without any further justification.
- Physical mapping
  - Use a physical situation to justify a mathematical result.
- Calculation
  - Rely on computational correctness – support from rules of math..
- Math consistency
  - Depend on similarity of math structure or “drilling down” into the math.

A novice / expert difference

- We regularly saw mid-level physics students "get stuck" in one epistemic mode for minutes at a time, missing useful warrants of other types.

- Experts tend to blend these modes together effectively and intertwine them.

Bing & Redish, Am. J. Phys. ***76*** *(2008) 418-424.*

Making meaning with math: Blending the physics and the math

- Our analysis suggests we need to rethink and unpack the structure of our knowledge
- Math plays a large role
in the upper division physics major.
- Rethink the role of mathematics
in problem solving.
  - Unpack to determine relevant (normative) E-Games
  - Understand what E-Games (ethnographic) students play

Unpacking math in physics

- Math in physics class is not the same as math in a math class
  - We use many different symbols — and not just in the standard “math” ways.
  - We use the same symbol to mean different things, the interpretation depending on context.
  - We blur the distinction between constants and variables depending on the physics.
  - We use equations not just to calculate but to organize our conceptual knowledge.
- But even more important – we put meaning to math differently from in a math class.

Traxoline

Attributed to Judith Lanier

- It is very important that you learn about traxoline. Traxoline is a new form of zionter. It is montilled in Ceristanna. The Ceristannians gristeriate large amounts of fevon and then bracter it to quasel traxoline. Traxoline may well be one of our most lukized snezlaus in the future because of our zionter lescelidge.

Directions: Answer the following questions in complete sentences. Use your best handwriting.

1. What is a traxoline?
2. Where is traxoline montilled?
3. How is traxoline quaselled?
4. Why is it important to know about traxoline?

Math-as-math and math-in-physics are different games!

What are we doing when we specify "the units" of a physical quantity?

We are determining which irreducible representation of the 3-parameter scaling group SxSxS it transforms by.

- Our fundamental processing of equations is more complex than in a math class.
  - We associate our interpretation of the equation with a physical system — which lends information on how to interpret the equation
  - We use particular symbols that carry ancillary information not otherwise present in the mathematical structure of the equation
  - We use more complex quantities than in math classes and use them tacitly.

An Example

- If

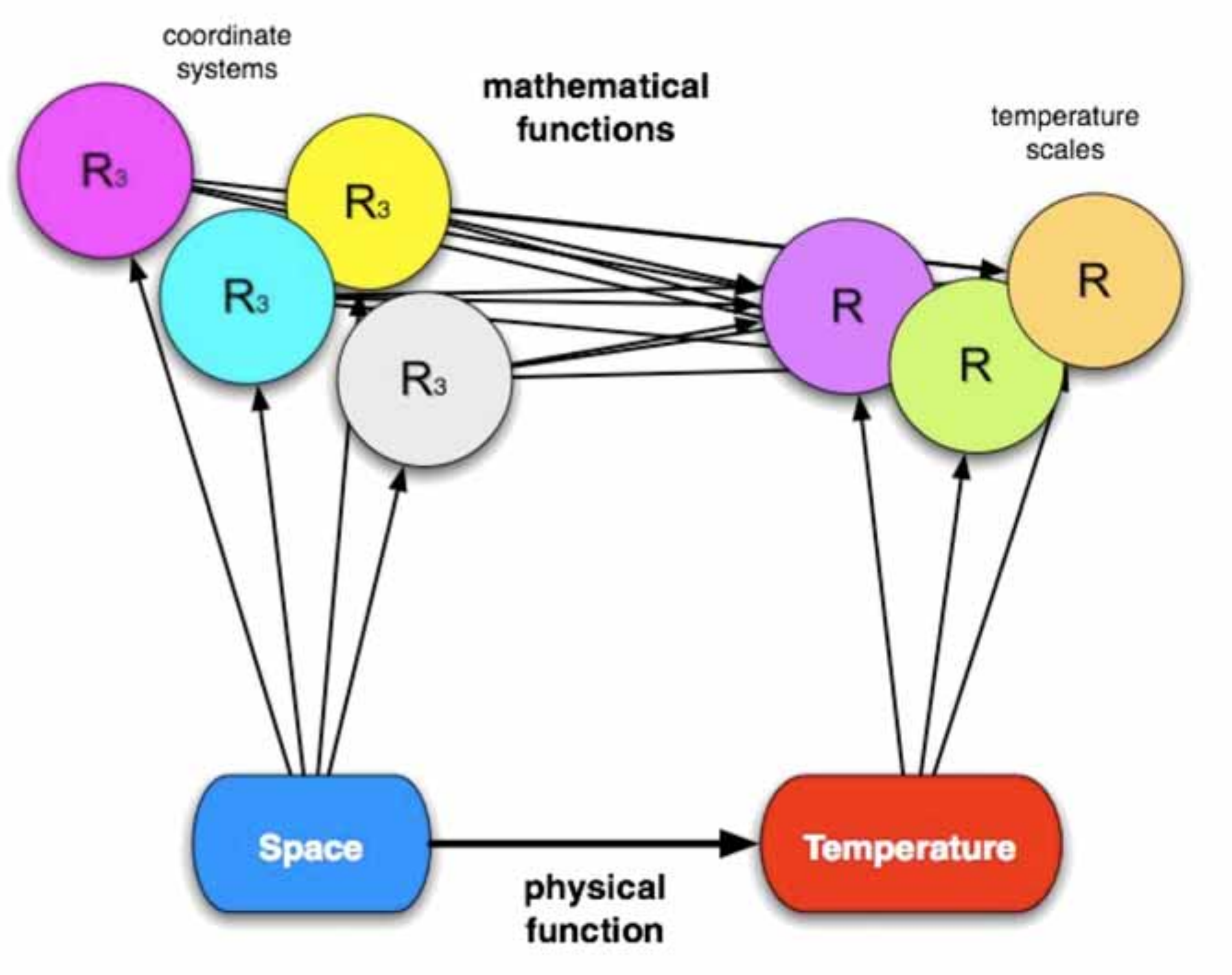
$$T(x,y) = k(x^2 + y^2)$$

$$where\ k\ is\ a\ constant$$

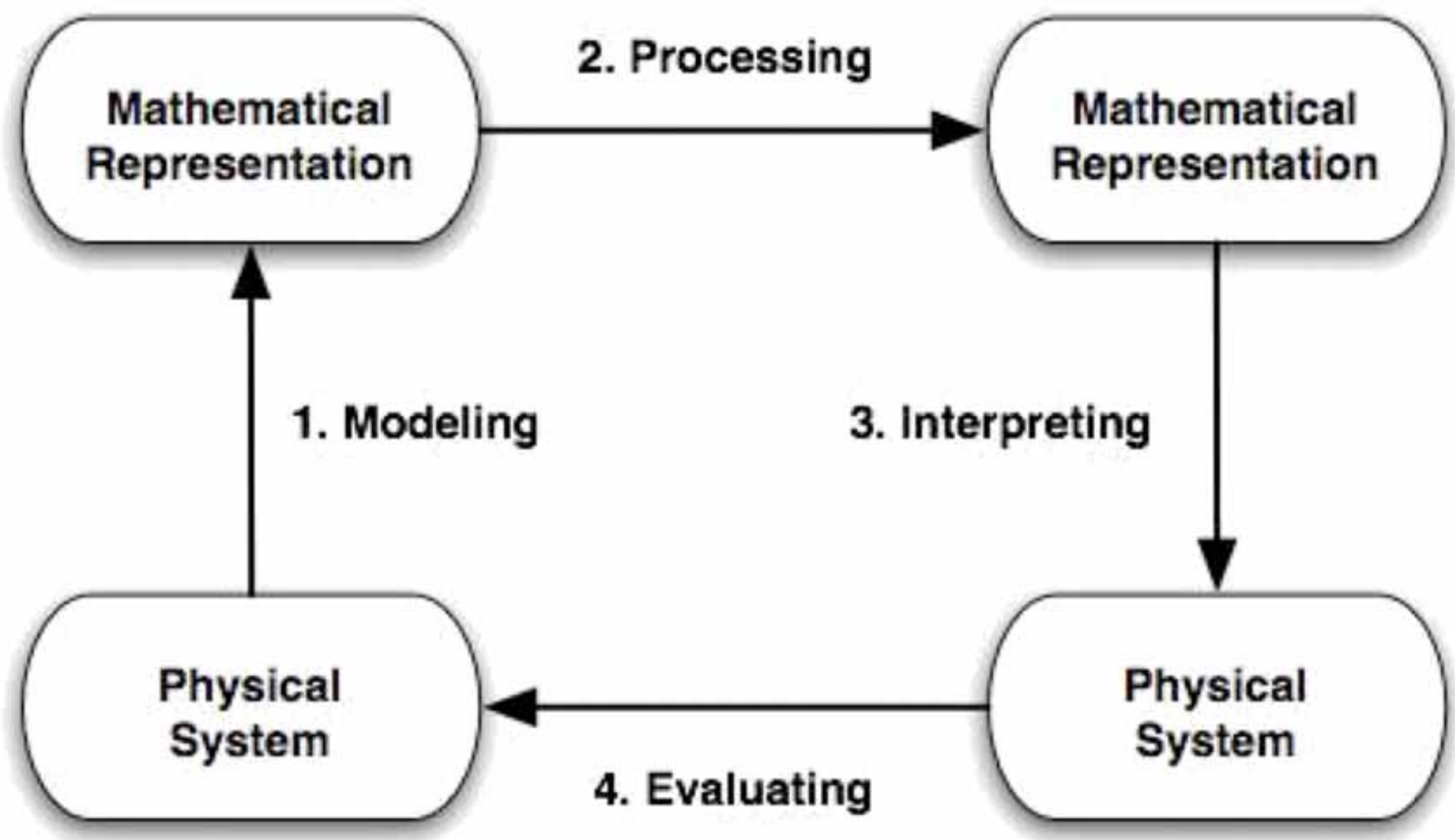
- then what is

$$T(r,\theta) = ?$$

Mathematical and physical functions are different



Unpacking our use of math in physics: Some useful games



Our traditional approach may not help students focus on critical issues.

- Texts and traditional problems often focus on processing and rarely ask students to model, interpret, or evaluate.
- Instructors may not be able to unpack their expert knowledge and fail to recognize what’s complex in a problem.
- Students don’t get these ideas in math and may ignore critical associations with physics even if the instructor discusses them.

A problem that goes beyond processing

The pair of coupled non-linear ODEs are referred to as the Lotka-Volterra equations and are supposed to represent the evolution of the populations of a predator and its prey in time.
The constants A, B, C, D are positive.

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = -Cy + Dxy$$

- *Which of the variables, x or y, represents the predator?*
- *Which represents the prey?*
- *What reasons do you have for your choice?*
- *What's left out of this model?*

Example: Undergrad QM

- The following problem was given in the second term of UG QM.
  - *A beam of electrons of energy E is incident on a square barrier of height V_0 and width a. Find the reflection and transmission coefficients, R and T.*
- The student in this example followed an expected procedure but was unable to recover from minor errors.

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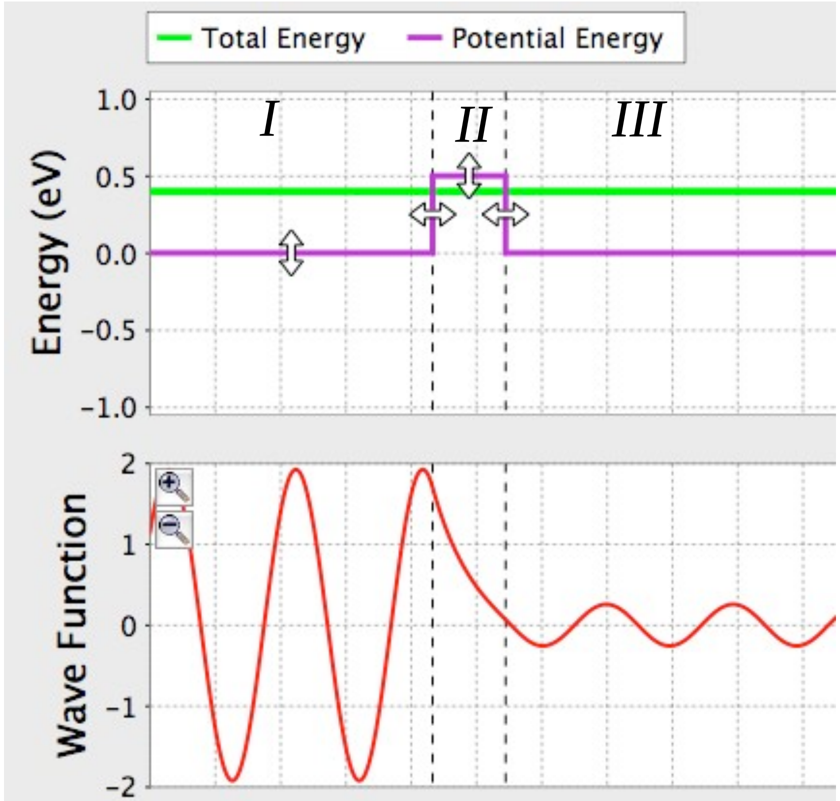


$$\psi_I = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II} = Ce^{\kappa x} + De^{-\kappa x}$$

$$\psi_{III} = Ee^{ikx} + Fe^{-ikx}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\kappa^2 = \frac{2m(V-E)}{\hbar^2}$$

* CU PhET simulation

$$\psi_I\big|_{x=0} = \psi_{II}\big|_{x=0} \qquad \psi_I'\big|_{x=0} = \psi_{II}'\big|_{x=0}$$

$$\psi_{II}\big|_{x=a} = \psi_{III}\big|_{x=a} \qquad \psi_{II}'\big|_{x=a} = \psi_{III}'\big|_{x=a}$$

4 equations in 6 unknowns

take $F = 0$

churn: solve for B, C, D, E in terms of A.

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|E|^2}{|A|^2}$$

A path to a solution?

- In class, the instructor showed the solutions in each region in lecture and the student had copied them down.
- He made some mistakes in copying – keeping the "i's" in the wave function's exponents in region II.
- He was totally stuck – kept looking through notes and text trying to find the "correct" form.
- He later showed that he was easily able to generate the solution from the SE.

What E-Game?

Professor

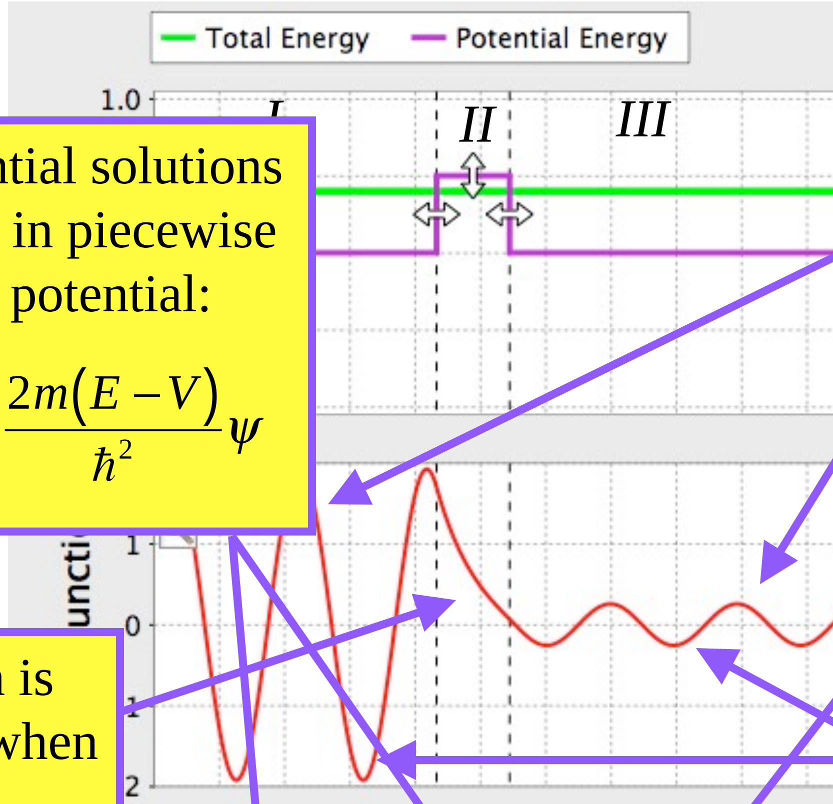
- Goal
  - Calculate R and T as function of E, V_0, a.
- Moves
  - Write ψ in each region.
  - Match ψ and ψ' across boundaries.
  - Find currents
  - Solve for R, T.
- Hidden moves
  - Check soln. with SE
  - Check units
  - …

Student

- Goal
  - Calculate R and T as function of E, V_0, a.
- Moves
  - Write ψ in each region.
  - Match ψ and ψ' across boundaries.
  - Find currents
  - Solve for R, T.
- Hidden moves
  - Copy solutions from lecture notes

Exponential solutions from SE in piecewise constant potential:

$$\frac{d^2\psi}{dx^2} = -\frac{2m(E-V)}{\hbar^2}\psi$$



*

Wavefunction oscillates when energy is positive

Change of sign of V-E leads to reversal of character of solutions.

Ampli... throug... barrier, but λ stays same (because E same on both sides)

Wavefunction is exponential when energy is negative

$$\psi_I = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II} = Ce^{\kappa x} + De^{-\kappa x}$$

$$\psi_{III} = Ee^{ikx} + Fe^{-ikx}$$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$\kappa^2 = \frac{2m(V-E)}{\hbar^2}$$

Growing exponential still present since region is bounded (doesn't go to infinity)

* CU PhET simulation

$$\psi_I\big|_{x=0} = \psi_{II}\big|_{x=0} \qquad \psi_I'\big|_{x=0} = \psi_{II}'\big|_{x=0}$$

$$\psi_{II}\big|_{x=a} = \psi_{III}\big|_{x=a} \qquad \psi_{II}'\big|_{x=a} = \psi_{III}'\big|_{x=a}$$

Continuous wavefunction and derivatives correspond to no infinite potentials.

4 equations in 6 unknowns

take $F = 0$

churn: solve for B, C, D, E in terms of A.

$$R = \frac{|B|^2}{|A|^2} \qquad T = \frac{|E|^2}{|A|^2}$$

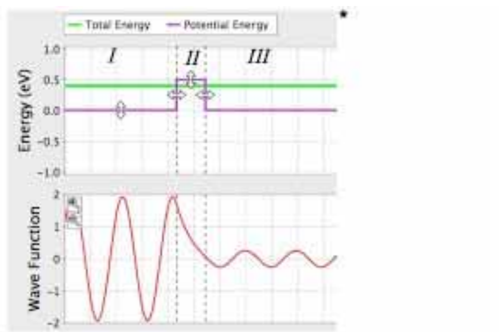
Treatment of solution relies on understanding of meaning of traveling waves.

Solutions relative to A, F must be 0.

Structure of coefficients depends on understanding of particle current.

Novice vs. Expert

- The novice solution gets the math
- The expert adds a web of physics associations



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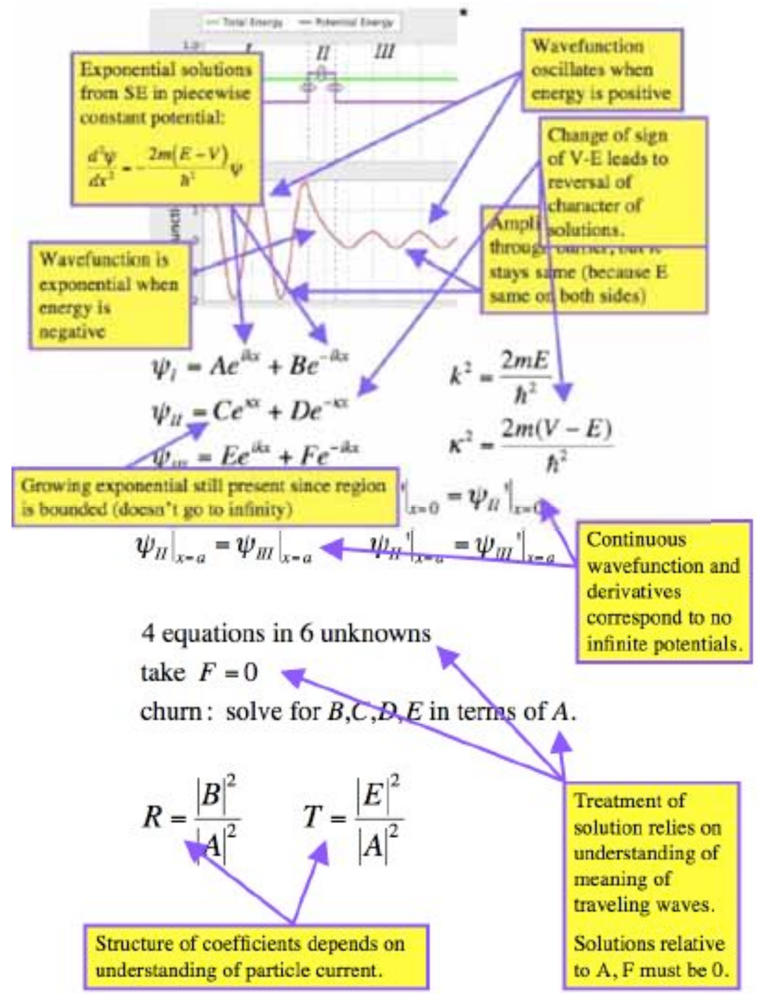

What good does all this do us?

- Fifty years ago, a great teacher said, “To be a great teacher, all you have to do is make things perfectly clear.”
- The problem is, “perfectly clear” lives in the head of the student, not of the teacher.
- As we learn, we forget what things look like to us as learners (compilation). We have to “reverse engineer” our own knowledge to see what it means to be clear. This takes work – and studying student thinking as well as our own.
- Understanding these issues helps us to
  - see that some “stupid mistakes” are not so stupid after all
  - understand our physics in new and deeper ways.

More information at

- *http://www.physics.umd.edu/perg*